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**Abstract :** In the beginning, geometry was a collection of rules for computing lengths, areas, and volumes. Many were crude approximations derived by trial and error. This body of knowledge was developed and used in construction, navigation, and surveying by the Babylonians and Egyptians.
The ancient knowledge of geometry was passed on to the Greeks. The Greek historian Herodotus (5th century BC) credits the Egyptians with having originated the subject, but there is much evidence that the Babylonians, the Hindu civilization, and the Chinese knew much of what was passed along to the Egyptians. Maybe we should say that the Greeks gathered all that they could find about geometry. They seemed to be blessed with an inclination toward speculative thinking and the leisure to pursue this inclination. They insisted that geometric statements can be established by deductive reasoning rather than trial and error. This began with Thales of Miletus (624-547 BC). He was familiar with the computations, right or wrong, handed down from Egyptian and Babylonian mathematics. In determining which of the computations were correct, he developed the first logical geometry. This orderly development of theorems by proof was the distinctive characteristic of Greek mathematics and was new. Their foundation of plane geometry was brought to a conclusion around 440 BC in the Elements by the mathematician Hippocrates of Chios (470-410 BC). This treatise has been lost, but many historians agree that it probably covered most of Books $I-IV$ of Euclid's Elements, which appeared about a century later, circa 300 BC. In this first Elements Hippocrates included geometric solutions to quadratic equations and early methods of integration. He studied the classic problem of squaring the circle showing how to square a lune. He worked on duplicating the cube which he showed equivalent to constructing two mean proportionals between a number and its double. Euclidean geometry was certainly conceived by its creators as an idealization of physical geometry. The entities of the mathematical system are concepts, suggested by, or abstracted from, physical experience but differing from physical entities as an idea of an object differs from the object itself. However, a remarkable correlation existed between the two systems. The angle sum of a mathematical triangle was stated to be 180, if one measured the angles of a physical triangle the angle sum did indeed seem to be 180, and so it went for a multitude of other relations. Because of this agreement between theory and practice, it is not surprising that many writers came to think of Euclid's axioms as self evident truths. Centuries later, the philosopher Immanuel Kant even took the position that the human mind is essentially Euclidean and can only conceive of space in Euclidean terms. Thus, almost from its inception, Euclidean geometry had something of the character of dogma. Euclid based his geometry on five fundamental assumptions :
**Postulate I :** For every point $P$ and for every point $Q$ not equal to $P$ there exists a unique line $\lambda$ that passes through P and $Q$.
**Postulate II :** For every segment $AB$ and for every segment $CD$ there exists a unique point $E$ such that $B$ is between $A$ and $E$ and segment $CD$ is congruent to segment $BE$.
**Postulate III :** For every point $O$ and every point $A$ not equal to $O$ there exists a circle with center $O$ and radius $OA$.

**Postulate IV :** All right angles are congruent to each other.
**Postulate V :** If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.
One of the themes that ran through Geometry following Euclid was the search for a replacement or a proof of dependence of his Fifth Postulate. No one seemed to like this Fifth Postulate, possibly not even Euclid himself. The reason that this statement seems out of place is that the first four postulates seem to follow from experience. The Fifth Postulate is non-intuitive. It does come from the study of parallel lines, though. An equivalent statement to this postulate is :
**Parallel Postulate :** "Given a line and a point not on that line, there exists one and only one line through that point parallel to the given line".

Since Euclid, geometry had meant the geometry of Euclidean space of two dimensions (plane geometry) or of three dimensions (solid geometry). The next tremendous advancement in the field of geometry occurred in the 17th century when René Descartes discovered coordinate geometry. Coordinates and equations could be used in this type of geometry in order to illustrate proofs. The creation of coordinate geometry opened the doors to the development of calculus and physics. Euclid's Fifth Postulate also called the parallel postulate seemed to be too burdensome. Since it is not intuitive, many thought it can be proved as a theorem following from this Axiom system, making it redundant In the first half of the nineteenth century there had been several developments complicating the picture. Mathematical applications required geometry of four or more dimensions ; the close scrutiny of the foundations of the traditional Euclidean geometry had revealed the independence of the parallel postulate from the others, and non-Euclidean geometry had been born. Carl Friedrich Gauss, Nikolai Lobachevsky, and János Bolyai formally discovered non-Euclidean geometry. In this kind of geometry, four of Euclid's first five postulates remained consistent, but the idea that parallel lines do not meet did not stay true. This idea is a driving force behind elliptical geometry and hyperbolic geometry. By 1872, non-Euclidean geometries had emerged, but without a way to determine their hierarchy and relationships. The Erlangen program is a method of characterizing geometries based on group theory and projective geometry. It was published by Felix Klein in 1872 as Vergleichende Betrachtungen über neuere geometrische Forschungen. It is named after the University Erlangen-Nürnberg, where Klein worked. Klein proposed an idea that all these new geometries are just special cases of the projective geometry, as already developed by Poncelet, Möbius, Cayley and others. With every geometry, Klein associated an underlying group of symmetries. The hierarchy of geometries is thus mathematically represented as a hierarchy of these groups, and hierarchy of their invariants. For example, lengths, angles and areas are preserved with respect to the Euclidean group of symmetries, while only the incidence structure and the cross-ratio are preserved under the most general projective transformations. A concept of parallelism, which is preserved in affine geometry, is not meaningful in projective geometry. Then, by abstracting the underlying groups of symmetries from the geometries, the relationships between them can be re-established at the group level. Since the group of affine geometry is a subgroup of the group of projective geometry, any notion invariant in projective geometry is a priori meaningful in affine geometry ; but not the other way round. If you add required symmetries, you have a more powerful theory but fewer concepts and theorems (which will be deeper and more general). In other words, the "traditional spaces" are homogeneous spaces ; but not for a uniquely determined group. Changing the group changes the appropriate geometric language. In today's language, the groups concerned in classical geometry are all very well known as Lie groups : the classical groups. The specific relationships are quite simply described, using technical language. For example, the group of projective geometry in n real-valued dimensions is the symmetry group of n-dimensional real projective space (the general linear group of degree $n+1$, quotiented by scalar matrices). The affine group will be the subgroup respecting (mapping to itself, not fixing pointwise) the chosen hyperplane at infinity. This subgroup has a known structure (semidirect product of the general linear group of degree n with the subgroup of translations). This description then tells us which properties are 'affine'. In Euclidean plane geometry terms, being a parallelogram is affine since affine transformations always take one parallelogram to another one. Being a circle is not affine since an affine shear will take a circle into an ellipse. To explain accurately the relationship between affine and Euclidean geometry, we now need to pin down the group of Euclidean geometry within the affine group. The Euclidean group is in fact (using the previous description of the affine group) the semi-direct product of the orthogonal (rotation and reflection) group with the translations. This gave birth to what we call Klein geometry, a type of geometry motivated by Felix Klein in his influential Erlangen program. More specifically, it is a homogeneous space $X$ together with a transitive action on $X$ by a Lie group $G$, which acts as the symmetry group of the geometry. A much more formal definition goes like this : A Klein geometry is a pair $(G, H)$ where $G$ is a Lie group and $H$ is a closed Lie subgroup of $G$ such that the (left) coset space $G/H$ is connected. The group $G$ is called the principal group of the geometry and $G/H$ is called the space of the geometry (or, by an abuse of terminology, simply the Klein geometry). The space $X = G/H$ of a Klein geometry is a smooth manifold of dimension

$$\dim\ X = \dim\ G - \dim\ H.$$

There is a natural smooth left action of $G$ on $X$ given by

$$g \cdot (aH) = (ga)H.$$

Clearly, this action is transitive (take $a = 1$), so that one may then regard $X$ as a homogeneous space for the action of $G$. The stabilizer of the identity coset $H \in X$ is precisely the group $H$. Given any connected smooth manifold $X$ and a smooth transitive action by a Lie group $G$ on $X$, we can construct an associated Klein geometry $(G, H)$ by fixing a basepoint $x_0$ in $X$ and letting $H$ be the stabilizer subgroup of $x_0$ in $G$. The group $H$ is necessarily a closed subgroup of $G$ and $X$ is naturally diffeomorphic to $G/H$. Two Klein geometries $(G_1, H_1)$ and $(G_2, H_2)$ are geometrically isomorphic if there is a Lie group isomorphism $\phi : G_1 \longrightarrow G_2$ so that $\phi(H_1) = H_2$. In particular, if $\phi$ is conjugation by an element $g \in G$, we see that $(G, H)$ and $(G, gHg^{-1})$ are isomorphic. The Klein geometry associated to a homogeneous space $X$ is then unique up to isomorphism (i.e. it is independent of the chosen basepoint $x_0$). As stated above, in October 1872, Felix Klein addressed at the University of Erlangen an inaugural lecture which has later become known and famous as the Erlangen Program. In this lecture, the importance of the term group for the classification of various geometries is elucidated. The whole Program consists of ten chapters. Fundamental ideas of Klein's classification of various geometries are presented in the first chapter where the following definition of such a geometry is stated :

*"Have a geometric space and some transformation group. A geometry is the study of those properties of the given geometric space that remain invariant under the transformations from this group. In other words, every geometry is the invariant theory of the given transformation group".*

Felix Klein emphasizes that the transformation group can be an arbitrary group. This definition served to codify essentially all the existing geometries of the time and pointed out the way how to define new geometries as well. Until that time various types of geometry, e.g. Euclidean, projective, hyperbolic, elliptic and so on, were all treated separately. Felix Klein set forth in his Program a unified conception of geometry that was far broader and more abstract than any one contemplated previously. At that time in Germany and elsewhere, much debate was going on about the validity of the recently developed non-Euclidean geometries. Felix Klein demonstrated in his Program that they could be modeled in projective geometry associated with Euclidean geometry. Since no one doubted the validity of Euclidean geometry, this important insight served to validate non-Euclidean geometries as well. In the second chapter of the Erlangen Program, Felix Klein defines an ordering of geometries in such a way that he transfers the inclusion relation among various transformation groups to the corresponding geometries. Replacing some transformation group by other transformation group in which the original group is involved, only a part of the former geometric properties remains invariant. The passage to a larger group or a subgroup of a transformation group makes it possible to pass from one type of geometry to another one. In this way the Erlangen Program codified a simple, but very important principle of ordering of particular geometries. To recap, all manifolds of dimension $n$ seem to be locally identical to $\mathbb{R}^n$ or $\mathbb{C}^n$. However, in terms of geometry, they are completely different. This justifies the fact that determining locally Euclidean spaces amounts to determining Clifford-Klein forms of Euclidean motion groups.
Motivated by the recently published works focusing on the study of deformation of discontinuous groups for homogeneous spaces, the team lead by Pr. Ali Baklouti is currently addressing this subject. Some answers are suggested in the cases of exponential, solvable and nilpotent Lie groups as well as Euclidean motion groups. Notably, those results seem to diverge : we cannot find a common formulation for them. Bizarrely, this latter divergence is rather expected and justified as every geometry has its own specificity.
In this context, the purpose of this dissertation is to study discontinuous actions in the case of compact extensions of $\mathbb{R}^n$. The second chapter presents an introductory section, in which we define and present different objects and tools that we are dealing with. The third chapter is devoted for establishing a general theorem of local rigidity. An application is given in the case of of compact extensions of $\mathbb{R}^n$. The fourth chapter is consecrated to the study of the structure of closed connected subgroups and and discrete subgroups of Euclidean motion groups. A criterion of proper action is therein established in the case of compact extensions of $\mathbb{R}^n$. As an application, we produce a necessary and sufficient condition for the existence of compact Clifford-Klein forms and we address when the Calabi-Markus phenomenon occurs. The last chapter is mainly an expository chapter which has two objectives : First, we pose some open questions in order to extend the study addressed within this manuscript in the case of nilpotent Lie groups and their compact extensions. Second, we gather and exhibit recent results that answer partially those questions.