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Abstract : The Russian mathematician S. N. Bernstein (one of the greatest approximation theorists of the last century) asked under which conditions on an increasing sequence $\Lambda = (0 = \lambda_0 < \lambda_1 < ...)$ one can guarantee that the vector space

$$\Pi(\Lambda) := \text{span}\{x^{\lambda_k}, k = 0, 1, ...\}$$

generated by the monomials x^{λ_k} is a dense subset of $C([0,1])$. Here $C([0,1])$ denotes the space of all continuous functions on $[0,1]$. He especially proved that the condition

$$\sum_{\lambda_k > 0} \frac{1 + \log \lambda_k}{\lambda_k} = \infty$$

is necessary and the condition

$$\lim_{k \to +\infty} \frac{\lambda_k}{k \log k} = 0$$

is sufficient, and conjectured that a necessary and sufficient condition to have $\overline{\Pi(\Lambda)} = C([0,1])$ is

$$\sum_{k=1}^{+\infty} \frac{1}{\lambda_k} = \infty.$$

As a simple consequence, the monomials $1, x, x^2, ...$ have dense linear span in $C([0,1])$. What, is so special about these particular powers ? How about if we consider polynomials of the form $\sum_{k=0}^{n} a_k x^{k^2}$; are they dense, too ? More generally, what can be said about the span of a sequence of monomials (x^{λ_n}), where $\lambda_0 < \lambda_1 < \lambda_2 < ...$? Of course, we will have to assume that $\lambda_0 \geq 0$, but it is not hard to see that we will actually need $\lambda_0 = 0$, for otherwise each of the polynomials $\sum_{k=0}^{n} a_k x^{\lambda_k}$ vanishes at $x = 0$ (and then has distance at least 1 from the constant 1 function, for example). If the λ_n are integers, it is also clear that we must have $\lambda_n \to \infty$ as $n \to \infty$. But what else is needed ? The answer comes to us from Müntz in 1914 (see [?]). We sometimes see the name Otto Szàsz associated with Müntz's theorem, because Szàsz proved a similar theorem nearly the same time (1916).

Theorem 0.1 *(Müntz, 1914) Let $\Lambda = (\lambda_i)_{i=0}^{\infty}$, $0 = \lambda_0 < \lambda_1 < ...$, be an increasing sequence of positive real numbers. Then $\Pi(\Lambda) = span\{x^{\lambda_k}, k = 0, 1, ...\}$, the Müntz space associated to Λ, is a dense subset of $C([0,1])$ if and only if*

$$\sum_{k=1}^{+\infty} \frac{1}{\lambda_k} = \infty.$$

This is a nice theorem because it connects a topological result (the density of a certain subset of a functional space) with an arithmetical one (the divergence of a ceratin harmonic series). Another reason to show the beauty of Müntz's theorem is that the original result does not only solve a typical problem but also opens the door to many new interesting questions. We can mention the example of some tests to change the space of continuous functions $C([0,1])$ to other function spaces such as $L^p(a,b)$, or to consider the analogous problem in several variables, on complex domains, on intervals away from the origin, for more general exponent sequences, for polynomials with integral coefficients. As a consequence, many proofs and generalizations of the theorem have been produced (For more details, we incite the reader to see [?]. It is shown that the interval $[0,1]$ in Müntz's theorem can be replaced by an arbitrary compact set $A \subset [0,+\infty[$ of positive Lebesgue measure. That is, if $A \subset [0,+\infty[$ is a compact set of positive Lebesgue measure, then $\mathrm{span}\{x^{\lambda_0}, x^{\lambda_1}, ...\}$ is dense in $C(A)$ if and only if $\sum_{j=1}^{+\infty} \frac{1}{\lambda_j} = +\infty$. Here $C(A)$ denotes the space of all real-valued continuous functions on A equipped with the uniform norm. If A contains an interval then this follows from the already mentioned results of Clarkson, Erdos and Schwartz. We can give a classical version of the Müntz-Szàsz theorem in $L^2([0,1])$ where $L^2([0,1])$ denotes the vector space of square integrable functions on $[0,1]$.

Theorem 0.2 *For $f \in L^2([0,1])$ and $\{n_k\}_{k=0}^{+\infty}$, a strictly increasing sequence of positive integers, we have :*

$$(1)\Big(\int_0^1 x^{n_k} f(x)dx = 0 \quad \text{for any} \;\; k \in \mathbb{N} \Longrightarrow f = 0 \;\; a.e.w\Big).$$

if and only if $\Big(\sum_{k=0}^{+\infty} \frac{1}{n_k} = +\infty\Big)$.

In the first chapter, my attention is focused on the generalization of this beautiful result to encompass some classes of Lie groups. In this case, it is not clear what the equivalent of monomials should be. In this context, Darwyn C. Cook considered a restrictive class of nilpotent Lie groups that have a fixed abelian polarizer for the open set of representations in general position. In these circumstances, the author produced a one-way Müntz-Szàsz analogue for the matrix coefficients of the operator valued Fourier transform.

The purpose here is to generate and prove several analogues of Theorem 0.2 for Euclidean motion groups $G = SO(n) \ltimes \mathbb{R}^n (n \geq 2)$ and for their universal coverings $\mathrm{spin}(n) \ltimes \mathbb{R}^n$. To do so, we are submitted to rephrase the condition of Theorem 0.2 above as an integral against a monomial of a family of coordinate functions depending upon the parameters involved in the spectrum of the Plancherel measure of G. We stated and proved a one way variant of Müntz-Szàsz's theorem. The motivation to seek another analogue comes from the fact that we look at a converse result.

The second chapter aims to tackle the context of compact extensions of Heisenberg groups. We propose a Müntz-Szàsz analogue for the matrix coefficients of the operator valued Fourier transform. The goal was to prove an analogous for the semi direct product $K \ltimes \mathbb{H}$, where K is a compact subgroup of automorphisms of the Heisenberg group $\mathbb{H}$. We have developed two variants of the above theorem in the context of compact extensions of Heisenberg groups. Here, it was necessary to use Müntz-Szàsz sequences in order to extend the classic Müntz-Szàsz theorem for functions with arbitrary support (functions with support not necessarily in $\mathbb{R}_+$). In particular, we treated the setting of Heisenberg groups.

Chapter 4 concerns some variants of Müntz-Szàsz theorem for a class of compact extensions of $\mathbb{R}^n$, which are semi direct products $K \ltimes \mathbb{R}^n$, where K designates a compact subgroup of automorphisms of $\mathbb{R}^n$. To achieve that, we thought firstly about looking for an analogue of Müntz-Szàsz's theorem for $K = SO_p(\mathbb{R}) \times SO_p(\mathbb{R})$, where $SO_p(\mathbb{R})$ designates the special orthogonal group and p, q are integers satisfying $p + q = n$.